

On The Eigenvalues of Some Vectorial Sturm-Liouville Eigenvalue Problems

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Abstract

The author tries to derive the asymptotic expression of the large eigenvalues of some vectorial Sturm-Liouville differential equations. A precise description for the formula of the square root of the large eigenvalues up to the $O(1/n)$ -term is obtained.

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1 Introduction

For the scalar case, it is well-known that: if $q^{(m)}(x) \in \mathcal{L}^1(0, \pi)$, the equation

$$-y'' + q(x)y = \lambda y, \quad y'(0) - hy(0) = 0, \quad y'(\pi) + Hy(\pi) = 0$$

has eigenvalues possessing the following asymptotic form

$$\sqrt{\lambda_n} = n + \frac{a_o}{n} + \cdots + \frac{a_{[m/2]}}{n^{2[m/2]+1}} + \frac{\gamma_n}{n^{2[m/2]+1}},$$

where $a_o = (h + H + 1/2 \int_0^\pi q(t)dt)/\pi$ and the other a_j , $1 \leq j \leq [m/2]$ are constant depending on $h, H, q(x)$ and the higher order derivatives of $q(x)$ up to order m . The main technique for deriving above expression is by using iterative method. One can find the standard approaches in the appendix in [9] or in [8]. For the vectorial case, the author only have seen analogous results in [3] for the case $d = 2$ and the boundary condtions are different to the cases the author has considered. Motivated by the arguments in [3], we still can use iterative method analogous to the one in [9], but only partial result as **Theorem 1** can be obtained. The technique we use in this section is similar to the one in [3], but the major difference between theirs and the author's is that the author locate the eigenvalues by using the associated matrix differential equation and by the theory of complex operator-valued functions. Hence, we can dispose any dimensional case. Consider the following vectorial Sturm-Liouville differential systems

$$-\phi''(x) + P(x)\phi(x) = \lambda\phi(x), \quad \phi'(0) + H_L\phi(0) = \mathbf{0} = \phi'(\pi) + H_R\phi(\pi) \quad (1)$$

where $P(x)$ is an $N \times N$ real symmetric matrix-valued function and H_L, H_R are $N \times N$ real symmetric constant matrices

Consider the matrix differential equations related to (1) and (FS), respectively,

$$\begin{aligned} -Y''(x) + P(x)Y(x) &= \lambda Y(x), \quad Y(0) = I, \quad Y'(0) = -H_L, \\ -\mathcal{C}''(x) &= \lambda \mathcal{C}(x), \quad \mathcal{C}(0) = I, \quad \mathcal{C}'(0) = -H_L, \end{aligned}$$

the solutions $Y(x; \lambda)$ and $\mathcal{C}(x; \lambda)$ are connected by the following identity

$$Y(x; \lambda) = \mathcal{C}(x; \lambda) + \int_0^x \mathcal{K}(x, t)\mathcal{C}(t; \lambda)dt, \quad \forall \lambda \quad (2)$$

and $\mathcal{C}(x; \lambda) = \cos(\sqrt{\lambda}x)I - \sin(\sqrt{\lambda}x)H_L/\sqrt{\lambda}$. The connection (2) is easy to prove, one can see a more general version in [4].

In this paper, we also use the symbol (P, H_L, H_R) to denote the eigenvalue problem (1). Denote $\Sigma(P, H_L, H_R)$ the sequence of eigenvalues of (P, H_L, H_R) . We can arrange those elements in it in ascending order as

$$\mu_0 \leq \mu_1 \leq \cdots \leq \mu_k \leq \cdots$$

We use the symbol $\sigma(P, H_L, H_R)$ to represent the set of eigenvalues of (P, H_L, H_R) , and arrange those elements in this set in ascending order as

$$\lambda_0 < \lambda_1 < \dots < \lambda_k < \dots$$

The multiplicity of each eigenvalues λ_k is denoted by m_k . For each positive integer n , an index set Λ_n is defined as

$$\Lambda_n = \{k : \lambda_k^{1/2} \in B_{1/4}(n)\}$$

where $B_r(\xi) = \{z : |z - \xi| < r\}$.

With above basic terminology and notations, we shall give the proof of the following theorem in section 2:

Theorem 1 *Suppose $\lambda_k \in \sigma(P, H_L, H_R)$ with multiplicity m_k . For n is sufficiently large, if $k \in \Lambda_n$, then*

$$\lambda_k^{1/2} = n + \frac{a_k}{n} + \gamma_n(k), \quad (3)$$

where a_k is the characteristic value of the $N \times N$ real symmetric matrix $\frac{1}{\pi}(\frac{1}{2} \int_0^\pi P(t)dt + H_R - H_L)$, and $\gamma_n(k) = o(1/n^2)$. Furthermore, $\sum_{k \in \Lambda_n} m_k = N$, where N is the dimension of the system (1).

2 Asymptotic analysis of eigenvalues

In order to make this section become more readable, the author outlines the sketch of the whole approach as below: The eigenvalues λ_k , $k \geq 0$ of (1) can be locate by determining whether the matrix-valued function

$$W(\lambda_k) = Y'(\pi; \lambda_k) + H_R Y(\pi; \lambda_k)$$

is singular or not. We use iterative method so as to write

$$W(\lambda) = \Psi(\lambda) + \mathcal{E}(\lambda) \quad (4)$$

where we shall see that $\Psi(\lambda)$ has a quite neat and simple form from which we can determine those values $\tilde{\lambda}$ making $\Psi(\lambda)$ be singular. By the extension theorem of Rouché's theorem on operator-valued functions, we are getting close to locate the eigenvalues of (1), although we can not locate them completely and explicitly as we did in the scalar case. With above short sketch in mind, we now start the lengthy and tedious computations on the asymptotics of eigenvalues of (1).

Rewrite

$$Y(x; \lambda) = \mathcal{C}(x; \lambda) + \int_0^x \frac{\sin \sqrt{\lambda}(x-t)}{\sqrt{\lambda}} P(t) Y(t; \lambda) dt. \quad (5)$$

Then

$$\begin{aligned} Y_1(x; \lambda) &= Y(x; \lambda) - \mathcal{C}(x; \lambda) \\ &= \int_0^x \frac{\sin \sqrt{\lambda}(x-t)}{\sqrt{\lambda}} P(t) \mathcal{C}(t; \lambda) dt + \int_0^x \frac{\sin \sqrt{\lambda}(x-t)}{\sqrt{\lambda}} P(t) Y_1(t; \lambda) dt \\ &= I_1 + \int_0^x \frac{\sin \sqrt{\lambda}(x-t)}{\sqrt{\lambda}} P(t) Y_1(t; \lambda) dt. \end{aligned}$$

Using integration by part, we can compute I_1 as

$$I_1 = \frac{\sin \sqrt{\lambda} x}{\sqrt{\lambda}} \mathcal{K}(x, x) + \frac{\cos \sqrt{\lambda} x}{\lambda} (\mathcal{K}(x, x) H_L + \frac{1}{4} (P(x) - P(0))) + O(|\lambda|^{-\frac{3}{2}}).$$

Combining with (6), we have

$$\begin{aligned} Y_1(x; \lambda) &= \frac{\sin \sqrt{\lambda} x}{\sqrt{\lambda}} \mathcal{K}(x, x) + \frac{\cos \sqrt{\lambda} x}{\lambda} (\mathcal{K}(x, x) H_L + \frac{1}{4} (P(x) - P(0))) \\ &\quad + \int_0^x \frac{\sin \sqrt{\lambda}(x-t)}{\sqrt{\lambda}} P(t) Y_1(t; \lambda) dt + O(|\lambda|^{-\frac{3}{2}}). \end{aligned}$$

As $|\Im \lambda| < \kappa$ for some constant κ fixed, by Gronwall's inequality, we can prove that $\|Y_1(x; \lambda)\|_\infty = O(|\lambda|^{-1/2})$, as $|\lambda|$ is sufficiently large. From above argument, we have

$$\begin{aligned} Y(x; \lambda) &= \cos \sqrt{\lambda} x (I + \frac{1}{\lambda} \mathcal{K}(x, x) H_L + \frac{1}{4\lambda} (P(x) - P(0))) \\ &\quad + \frac{\sin \sqrt{\lambda} x}{\sqrt{\lambda}} (\mathcal{K}(x, x) - H_L) + O(|\lambda|^{-\frac{3}{2}}). \end{aligned} \quad (7)$$

On the other hand, from (5),

$$Y'(x; \lambda) = \mathcal{C}'(x; \lambda) + \int_0^x \cos \sqrt{\lambda}(x-t) P(t) Y(t; \lambda) dt. \quad (8)$$

Plugging (7) into (8), and repeating the analysis as above, we have

$$\begin{aligned} Y'(x; \lambda) &= -\sqrt{\lambda} \sin \sqrt{\lambda} x I + \cos \sqrt{\lambda} x (\mathcal{K}(x, x) - H_L) \\ &\quad + \frac{\sin \sqrt{\lambda} x}{\sqrt{\lambda}} \left(\frac{P(x) + P(0)}{2} + \frac{1}{2} \int_0^x P(t) \mathcal{K}(t, t) dt - \mathcal{K}(x, x) H_L \right) \\ &\quad + \frac{\cos \sqrt{\lambda} x}{\lambda} \left(\frac{1}{4} (P'(x) - P'(0)) + \frac{1}{2} \left(\int_0^x P(t) \mathcal{K}(t, t) dt \right) H_L \right. \\ &\quad \left. + \frac{1}{8} \int_0^x (P^2(t) - P(t) P(0)) dt + \frac{1}{2} (P(x) \mathcal{K}(x, x) - H_L) \right) + O(|\lambda|^{-\frac{3}{2}}) \end{aligned} \quad (9)$$

By (7) and (9), we have

$$W(\lambda) = -\sqrt{\lambda} \sin \sqrt{\lambda} \pi I + \cos \sqrt{\lambda} \pi \mathcal{G}_1 + \frac{\sin \sqrt{\lambda} \pi}{\sqrt{\lambda}} \mathcal{G}_2 + \frac{\cos \sqrt{\lambda} \pi}{\lambda} \mathcal{G}_3 + O(|\lambda|^{-\frac{3}{2}}), \quad (10)$$

where

$$\mathcal{G}_1 = H_R - H_L + \mathcal{K}(\pi, \pi),$$

$$\begin{aligned} \mathcal{G}_2 &= \frac{P(\pi) + P(0)}{2} + \frac{1}{2} \int_0^\pi P(t) \mathcal{K}(t, t) dt + H_R \mathcal{K}(\pi, \pi) \\ &\quad - \mathcal{K}(\pi, \pi) H_L - H_R H_L, \end{aligned}$$

and

$$\begin{aligned} \mathcal{G}_3 &= \frac{P'(\pi) - P'(0)}{4} + \frac{1}{2} \left(\int_0^\pi P(t) \mathcal{K}(t, t) dt \right) H_L + \frac{1}{4} H_R (P(\pi) - P(0)) \\ &\quad + \frac{1}{8} \int_0^\pi (P^2(t) - P(t)P(0)) dt + \frac{1}{2} (P(\pi) \mathcal{K}(\pi, \pi) - H_L) + H_R \mathcal{K}(\pi, \pi) H_L \end{aligned}$$

Write $\lambda = \mu^2$. Then the term $\Psi(\lambda)$ in (4) is defined as $\Psi(\mu^2) = -\mu \sin \mu \pi I + \cos \mu \pi \mathcal{G}_1$. The remainder terms of (10) is denoted by $\mathcal{E}(\mu^2)$. Denote $\varpi(\mu) = \det(W(\mu^2))$ and $\mu = \sigma + it$. By the following identities

$$\begin{aligned} \sin(\sigma \pi + it \pi) &= \sin \sigma \pi \cosh t \pi + i \sinh t \pi \cos \sigma \pi, \\ \cos(\sigma \pi + it \pi) &= \cos \sigma \pi \cosh t \pi - i \sinh t \pi \sin \sigma \pi \end{aligned}$$

and using the Laplace expansion of determinants, we have the following result: for any complex μ , $\varpi(\mu) = \vartheta(\mu) + \varepsilon(\mu)$ where $\vartheta(\mu) = \mu^N \sin^N \mu \pi$ and $|\varepsilon(\mu)/\vartheta(\mu)| = O(1/|\mu|)$. With above facts and Rouché's theorem, there lie exactly N zeroes of $\varpi(\mu)$ within a suitable neighborhood of any sufficiently large integer n . If we denote a zero of $\varpi(\mu)$ by $n + \delta_{ni}$, it is not difficult to see that $\delta_{ni} = O(1/n)$. In fact, we can compute $\lim_{n \rightarrow \infty} \delta_{ni}$, and hence we have the first part of **Theorem 1**.

To touch down the goal, let us back to (4). The matrix-valued function $\Psi(\mu)$ actually provides us much information. Since the matrix \mathcal{G}_1 is real symmetric, there exists an orthogonal matrix U , $U^{-1} = U^*$, such that

$$U^* \Psi(\mu) U = \text{diag}[\varrho_1(\mu), \varrho_2(\mu), \dots, \varrho_N(\mu)]$$

where $\varrho_i(\mu) = -\mu \sin \mu \pi + \sigma_i \pi \cos \mu \pi$, $1 \leq i \leq N$, and σ_i , $1 \leq i \leq N$ are those characteristic values of constant matrix \mathcal{G}_1 . By (4) and (10), we have

$$\begin{aligned} U^* W(\mu^2) U &= U^* \Psi(\mu^2) U + U^* \mathcal{E}(\mu^2) U \\ &= U^* \Psi(\mu^2) U + \frac{\sin \mu \pi}{\mu} U^* \mathcal{G}_2 U + O\left(\frac{1}{\mu^2}\right). \end{aligned} \quad (11)$$

Using the following Rouché's Theorem for operator-valued functions in [7], we can suitably locate the eigenvalues of (1) from (11).

Theorem 2 *Let $W(\mu)$ and $S(\mu)$ be $\mathcal{L}(\mathcal{H})$ -valued holomorphic function defined on a region Ω enclosed by a contour Γ . Suppose W is normal with respect to Γ and let $V = W + S$. If*

$$\|W^{-1}(\mu)S(\mu)\| < 1, \quad (12)$$

then V is also normal with respect to Γ , and

$$m(\Gamma; V(\cdot)) = m(\Gamma; W(\cdot)).$$

One can refer to [7, chapter XI, §9, 205–211] for the definition of all the notations in above theorem. In our case, the first task we must face is the estimate of the norm of $U^*\Psi^{-1}(\mu)\mathcal{E}(\mu)U$ on some suitable contours. The estimate will be taken with respect to ∞ -norm, but this task is very tedious, we put the details in the appendix and give only the statement as below here. In the following lemma, the contour is chosen as: for any complex number μ_o ,

$$\Gamma_n(\mu_o) = \{\mu : |\mu - \mu_o| = O(\frac{1}{n^2})\} \quad (13)$$

in which we do not specify the radius since we shall obtain a uniform estimate.

Lemma 3 *Let μ_o be a zero of the transcendental equation*

$$\mu \sin \mu\pi - \alpha\pi \cos \mu\pi = 0,$$

locating in a suitable neighborhood of some sufficiently large integer n , where $\alpha\pi$ is a characteristic value of \mathcal{G}_1 . Let $\Gamma_n(\mu_o)$ is chosen as (13) with the same n . Then

$$\|U^*\Psi^{-1}(\mu)\mathcal{E}(\mu)U\|_\infty = O(\frac{1}{n}), \quad \forall \mu \in \Gamma_n(\mu_o).$$

Using Lagrange inversion formula, it is not difficult to see that the zero of the transcendental equation

$$\mu \sin \mu\pi - \alpha\pi \cos \mu\pi = 0,$$

lying in a suitable neighborhood of positive integer n can be expressed as

$$\mu_n = n + \frac{\alpha}{n} + \frac{\kappa(\alpha)}{n^3} + \dots$$

By **Theorem 2** and **Lemma 3**, if $\alpha\pi$ is a characteristic value of \mathcal{G}_1 , then there exist at least one eigenvalue $\lambda_n(\alpha)$ of (1) whose square root $\sqrt{\lambda_n(\alpha)}$ satisfies

$$\sqrt{\lambda_n(\alpha)} = n + \frac{\alpha}{n} + \gamma_n(\alpha), \quad (14)$$

where $\gamma_n(\alpha) = o(\frac{1}{n^2})$. We summarize above as

Lemma 4 Suppose the eigenvalues of (1) satisfying (14) in asymptotic sense. Then

$$\{\alpha\pi : \text{all possible } \alpha \text{ that (14) holds} \} \supseteq \sigma(\mathcal{G}_1),$$

where $\sigma(\mathcal{G}_1)$ is the spectral set of \mathcal{G}_1 .

On the other hand, we have the following result.

Lemma 5 If any sequence of eigenvalue of (1) has the asymptotic expression as (14), then $\alpha\pi \in \sigma(\mathcal{G}_1)$ and $\gamma_n(\alpha) = o(1/n^2)$.

Proof. By (10),

$$W(\mu^2) = -\mu \sin \mu\pi I + \cos \mu\pi \mathcal{G}_1 + \frac{\sin \mu\pi}{\mu} \mathcal{G}_2 + \frac{\cos \mu\pi}{\mu^2} \mathcal{G}_3 + O\left(\frac{1}{|\mu|^3}\right). \quad (15)$$

Let $\mu_n = \sqrt{\lambda_n}(\alpha)$ as given by (14), then plugging μ_n into (15), we have

$$\begin{aligned} W(\mu_n^2) &= -(n + \frac{\alpha}{n} + \gamma_n(\alpha)) \sin(n + \frac{\alpha}{n} + \gamma_n(\alpha))\pi I + \cos(n + \frac{\alpha}{n} + \gamma_n(\alpha))\pi \mathcal{G}_1 \\ &\quad + \frac{1}{n + \frac{\alpha}{n} + \gamma_n(\alpha)} \sin(n + \frac{\alpha}{n} + \gamma_n(\alpha))\pi \mathcal{G}_2 \\ &\quad + \frac{1}{(n + \frac{\alpha}{n} + \gamma_n(\alpha))^2} \cos(n + \frac{\alpha}{n} + \gamma_n(\alpha))\pi \mathcal{G}_3 + O\left(\frac{1}{n^3}\right). \end{aligned} \quad (16)$$

Since

$$\begin{aligned} \sin(\mu_n\pi) &= (-1)^n \left(\frac{\alpha}{n} + \gamma_n(\alpha)\right)\pi + O\left(\frac{1}{n^3}\right), \\ \cos(\mu_n\pi) &= (-1)^n \left(1 - \frac{1}{2}\left(\frac{\alpha}{n} + \gamma_n(\alpha)\right)^2 \pi^2\right) + O\left(\frac{1}{n^4}\right). \end{aligned}$$

Substituting them into (16) and equating the terms by comparing orders, we have

$$W(\mu_n^2) = (-1)^n [\alpha\pi I - \mathcal{G}_1 + n\gamma_n(\alpha)\pi I] + O\left(\frac{1}{n^2}\right). \quad (17)$$

Since $W(\mu_n^2)$ is singular for all n sufficiently large, and $n\gamma_n(\alpha) \rightarrow 0$, by the continuity of determinant function, the matrix $\alpha\pi I - \mathcal{G}_1$ must be singular. Otherwise, letting $\mathcal{T} = \alpha\pi I - \mathcal{G}_1$, we have, by (17), that

$$\|W(\mu_n^2) - (-1)^n \mathcal{T}\| \leq |n\gamma_n(\alpha)\pi| + O\left(\frac{1}{n^2}\right).$$

Thus if \mathcal{T} is invertible, then so is $W(\mu_n^2)$ for n sufficiently large, which is absurd. Therefore, $\alpha\pi$ is a characteristic value of \mathcal{G}_1 .

By **Lemma 4** and **Lemma 5**, we have **Theorem 1**.

Remark. Above approaches can be extended to some other type of boundary conditions, the author has disposed some kinds of them in [5], e.g., the Dirichlet boundary conditions. The determination on the first term in (3) is quite precise, while on the second or higher terms, there are still some ambiguities need to be excluded.

Appendix. The estimate of $\|U^*\Psi^{-1}(\mu)\mathcal{E}(\mu)U\|_\infty$

In section 2, we give the statement on the estimate of $\|U^*\Psi^{-1}(\mu)\mathcal{E}(\mu)U\|_\infty$. The proof of that lemma lies on the estimate of orders on the prescribed contour.

By (10) and (11),

$$\begin{aligned} U^*\Psi^{-1}(\mu)\mathcal{E}(\mu)U &= \mathcal{R}(\mu)\left(\frac{\sin \mu\pi}{\mu}U^*\mathcal{G}_2U + \frac{\cos \mu\pi}{\mu^2}U^*\mathcal{G}_3U\right) \\ &\quad + \mathcal{R}(\mu)O\left(\frac{1}{|\mu|^3}\right), \end{aligned}$$

where $\mathcal{R}(\mu) = \text{diag}[\varrho_1(\mu), \dots, \varrho_N(\mu)]$. Let μ_o be a value making $\Psi(\mu_o)$ be singular. Then μ_o is a zero of $\varrho_j(\mu)$ for some j . According to the distribution of the zeroes of the transcendental equation $\varrho_j(\mu)$, we may assume μ_o lie in a suitable neighborhood of a sufficient large integer n , and can be expressed as

$$\mu_o = n + \frac{a}{n} + \frac{b}{n^3} + O\left(\frac{1}{n^5}\right),$$

where b depends on a . With this kind μ_o as center, we choose $\Gamma_n(\mu_o)$ as defined in (13). We shall show that: On such contours, the two quantities

$$\|\mathcal{R}(\mu)\frac{\sin \mu\pi}{\mu}\|_\infty, \quad \|\mathcal{R}(\mu)\frac{\cos \mu\pi}{\mu^2}\|_\infty$$

are of order $O(1/n)$, and hence the remainder term is of order less than $O(1/n)$.

Here, we only give the demonstration on the first quantity. As for the other one, one can use the arguments in the coming paragraphs to prove it.

Write

$$\mathcal{R}(\mu)\frac{\sin \mu\pi}{\mu} = \begin{pmatrix} \chi_1(\mu) & 0 & \cdots & 0 \\ 0 & \chi_2(\mu) & \cdots & \vdots \\ \vdots & \vdots & \ddots & 0 \\ 0 & \cdots & 0 & \chi_N(\mu) \end{pmatrix}$$

where $\chi_j(\mu) = \sin \mu\pi / \mu(\mu \sin \mu\pi - \sigma_j\pi \cos \mu\pi)$, $1 \leq j \leq N$. Then

$$\|\mathcal{R}(\mu)\frac{\sin \mu\pi}{\mu}\|_\infty = \max_{1 \leq j \leq N} \|\chi_j(\mu)\|_\infty = \max_{1 \leq j \leq N} \sup_{\mu \in \Gamma_n(\mu_o)} |\chi_j(\mu)|.$$

We only to give the estimate on $|\chi_j(\mu)|$ in the sense of growth of order. Write $\mu = s + it$, then for any $\mu \in \Gamma_n(\mu_o)$, we have

$$\mu = n + \frac{a}{n} + \frac{\delta \cos \phi}{n^2} + i \frac{\delta \sin \phi}{n^2} + \frac{b}{n^3} + O\left(\frac{1}{n^5}\right),$$

where δ is the radius of the contour and $0 \leq \phi \leq 2\pi$. Then,

$$s = n + \frac{a}{n} + \frac{\delta \cos \phi}{n^2} + \frac{b}{n^3} + O\left(\frac{1}{n^5}\right), \quad (18)$$

$$t = \frac{\delta \sin \phi}{n^2}. \quad (19)$$

The following identities are needed:

$$|\sin \mu\pi|^2 = \frac{1}{2}(\cosh 2t\pi - \cos 2s\pi),$$

$$|\cos \mu\pi|^2 = \frac{1}{2}(\cosh 2t\pi + \cos 2s\pi),$$

$$2\Re\{\mu \sin \mu\pi \cos \bar{\mu}\pi\} = s \sin 2s\pi + t \sinh 2t\pi,$$

and

$$\begin{aligned} |\mu \sin \mu\pi - \theta\pi \cos \mu\pi|^2 &= |\mu \sin \mu\pi|^2 \\ &+ (\theta\pi)^2 |\cos \mu\pi|^2 - 2\theta\pi \Re\{\mu \sin \mu\pi \cos \bar{\mu}\pi\}, \end{aligned}$$

where $\theta \in \mathbf{R}$. Plugging (18), (19) into above identities, we have

$$\begin{aligned} |\sin \mu\pi|^2 &= \frac{a^2\pi^2}{n^2} + \frac{2\pi^2}{n^3}a\delta \cos \phi \\ &+ \frac{\pi^2}{3n^4}(-a^4\pi^2 + 6ab + 3\delta^2) + O\left(\frac{1}{n^5}\right), \quad \text{if } a \neq 0, \end{aligned}$$

$$|\sin \mu\pi|^2 = \frac{\pi^2\delta^2}{n^4} + \frac{\pi^4}{3n^8}\delta^4(1 - 2\cos^2 \phi) + O\left(\frac{1}{n^9}\right), \quad \text{if } a = 0.$$

$$\begin{aligned} |\cos \mu\pi|^2 &= 1 - \frac{a^2\pi^2}{n^2} - \frac{2\pi^2}{n^3}a\delta \cos \phi \\ &+ \frac{\pi^2}{3n^4}(3\delta^2(a^4\pi^2 - 6ab + 1 - 2\cos^2 \phi)) + O\left(\frac{1}{n^6}\right), \quad \text{if } a \neq 0, \end{aligned}$$

$$|\cos \mu\pi|^2 = 1 + \frac{\pi^2}{n^4}(1 - 2\cos^2 \phi) + O\left(\frac{1}{n^8}\right), \quad \text{if } a = 0.$$

Hence we have :

$$|\mu \sin \mu\pi|^2 + (\theta\pi)^2 |\cos \mu\pi|^2 = (\theta^2 + a^2)\pi^2 + \frac{2\pi^2}{n}a\delta \cos \phi \\ + \frac{\pi^2}{3n^2}(3\delta^2 - a^4\pi^2 + 6a^3 - 3(\pi a\theta)^2 + 6ab) + O(\frac{1}{n^3}), \quad \text{if } a \neq 0,$$

$$|\mu \sin \mu\pi|^2 + (\theta\pi)^2 |\cos \mu\pi|^2 = (\theta\pi)^2 + \frac{\pi^2\delta^2}{n^2} + O(\frac{1}{n^4}), \quad \text{if } a = 0.$$

On the other hand, we have

$$2\Re\{\mu \sin \mu\pi \cos \bar{\mu}\pi\} = 2a\pi + \frac{2\pi}{n}\delta \cos \phi \\ + \frac{2\pi}{3n^2}(3a^2 + 3b - 2\pi^2a^3) + O(\frac{1}{n^3}), \quad \text{if } a \neq 0,$$

$$2\Re\{\mu \sin \mu\pi \cos \bar{\mu}\pi\} = \frac{2\pi}{n}\delta \cos \phi + O(\frac{1}{n^4}), \quad \text{if } a = 0.$$

Therefore, we obtained

$$|\mu \sin \mu\pi - \theta\pi \cos \mu\pi|^2 \\ = (a\pi - \theta\pi)^2 + \frac{2\pi^2}{n}\delta(a - \theta) \cos \phi + \frac{\pi^2}{3n^2}(6(a - \theta)(a^2 + b) \\ + 3\delta^2 - \pi^2a^2(a^2 + 3\theta^2 - 4a\theta)) + O(\frac{1}{n^3}), \quad \text{if } a \neq 0,$$

$$|\mu \sin \mu\pi - \theta\pi \cos \mu\pi|^2 = (\theta\pi)^2 - \frac{2\pi^2}{n}\theta \cos \phi + \frac{\pi^2\delta^2}{n^2} + O(\frac{1}{n^4}), \quad \text{if } a = 0.$$

Finally, we have : On $\Gamma_n(\mu_\circ)$,

$$|\sin \mu\pi| = \frac{1}{n}|a|\pi + o(\frac{1}{n}), \quad \text{if } a \neq 0, \\ |\sin \mu\pi| = \frac{1}{n^2}\delta\pi + o(\frac{1}{n^2}), \quad \text{if } a = 0.$$

and

$$|\mu \sin \mu\pi - \theta\pi \cos \mu\pi| = O(1), \quad \text{if } \theta \neq a. \\ |\mu \sin \mu\pi - \theta\pi \cos \mu\pi| = \frac{1}{n}\delta\pi + o(\frac{1}{n}), \text{ if } a \neq 0, \text{ if } \theta = a. \\ |\mu \sin \mu\pi - \theta\pi \cos \mu\pi| = \frac{1}{n}\delta\pi + o(\frac{1}{n}), \text{ if } a = 0, \text{ if } \theta = a.$$

So, if $a \neq 0$, we obtain

$$\begin{aligned} |\chi_j(\mu)| &= \frac{\frac{1}{n}|a|\pi + o(\frac{1}{n})}{(n + o(1))(|\theta - a| + o(1))} = O(\frac{1}{n^2}), \quad \text{if } \theta \neq a. \\ |\chi_j(\mu)| &= \frac{\frac{1}{n}|a|\pi + o(\frac{1}{n})}{(n + o(1))(\frac{1}{n}\pi\delta + o(\frac{1}{n}))} = \frac{1}{n\delta}|\alpha\pi| + o(\frac{1}{n}), \quad \text{if } \theta = a. \end{aligned}$$

while $a = 0$,

$$\begin{aligned} |\chi_j(\mu)| &= \frac{\frac{1}{n^2}\delta\pi + o(\frac{1}{n^2})}{(n + o(1))(|\theta| + o(1))} = \frac{1}{n^3} \frac{\delta\pi}{|\theta|}, \quad \text{if } \theta \neq a, \\ |\chi_j(\mu)| &= \frac{\frac{1}{n^2}\delta\pi + o(\frac{1}{n^2})}{(n + o(1))(\frac{1}{n}\delta\pi + o(\frac{1}{n}))} = \frac{1}{n^2} + o(\frac{1}{n^2}), \quad \text{if } \theta = a. \end{aligned}$$

Summarize above results as : for n sufficiently large,

$$\begin{aligned} \|\mathcal{R}(\mu) \frac{\sin \mu\pi}{\mu}\|_{\infty} &\sim \frac{1}{\delta n} |a|\pi, \quad \text{if } a \neq 0, \\ \|\mathcal{R}(\mu) \frac{\sin \mu\pi}{\mu}\|_{\infty} &\sim O(\frac{1}{n^2}), \quad \text{if } a = 0. \end{aligned}$$

This tells us that $\|\mathcal{R}(\mu) \frac{\sin \mu\pi}{\mu}\|_{\infty}$ has growth order at most $O(1/n)$ as n is sufficiently large. With such estimate on the contours, we have the desired estimate on $\|U^* \Psi^{-1}(\mu) \mathcal{E}(\mu) U\|$, so we complete the proof **Lemma 3**.

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